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REVIEW OF NATURE-INSPIRED POSITION-DEPENDENT MASS OSCILLATOR MODELS APPLIED IN QUANTUM SYSTEMS

ABSTRACT: This review explores position-dependent mass (PDM) oscillators, naturally occurring in biological systems and relevant to engineered quantum devices. In nature, PDM oscillations appear in the bending of plant stems, fish swimming, bird flight, and the motion of limbs, as well as in the oscillatory behavior of organs like the heart and vocal cords. These phenomena inspired models of oscillators whose effective mass varies with position, simulating elastic structures attached to bodies of variable mass. Mathematically, such systems are captured by Liénard equations with quadratic velocity terms. This review examines key features – motion type, period, and amplitude – of PDM oscillators, highlighting their versatility for describing spatially varying inertia and dynamic adaptation. Extending these concepts to quantum systems, spatial variations in carrier mass arise in semiconductor nanostructures like quantum wells, wires, and dots, due to compositional inhomogeneities and structural gradients. Position-dependent mass models refine quantum mechanics, enabling more accurate energy-level and carrier-dynamics predictions. Such models are central to the design of advanced electronic and photonic devices, including quantum cascade lasers, high electron mobility transistors, and scanning tunneling microscopy. Bridging biology and quantum engineering, PDM oscillators offer a robust framework for innovation in adaptive materials and biologically inspired technologies. Future research should address nonlinear effects, anisotropic materials, and leverage data-driven optimization to fully realize the technological potential of PDM oscillators.

KEYWORDS: Position-dependent mass, nature inspired models, nonlinear oscillators, quantum systems

INTRODUCTION

Recent advances in nanotechnology and semiconductor physics have increasingly highlighted the significance of position-dependent mass (PDM) in

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quantum systems. In particular, charge carriers in engineered nanostructures, such as quantum wells, wires, and dots, often experience spatially varying effective masses due to material heterogeneity, strain, and compositional gradients. Traditional quantum mechanical models must therefore be refined to account for these variations, leading to the development of PDM frameworks. Remarkably, the principles of PDM are not confined to the realm of quantum physics. A significant number of PDM oscillatory systems are evident in the natural world, from the bending of tree branches in the wind to the undulatory locomotion of fish and the dynamic shape changes in insect wings. These biological systems adapt their mass distribution to optimize movement and resilience, offering compelling analogies for both mechanical and quantum PDM oscillators.

This paper aims to bridge these natural inspirations and their theoretical interpretations in quantum systems. The authors first explore the occurrence of PDM oscillators in living systems, highlighting how nature leverages spatial mass variation for enhanced function. Afterwards, a comprehensive theoretical framework for PDM models is developed, inspired by these biological phenomena. Finally, the authors discuss the mechanical-quantum analogy and the applications of PDM models in advanced materials and devices.

The remainder of this paper is organized as follows: section 2 examines nature-inspired PDM oscillators across plant and animal systems, section 3 details the theoretical framework of PDM models, including their mathematical formulations and physical interpretations, section 4 focuses on the mechanical analogy and the cross-disciplinary relevance of PDM oscillators and quantum applications, section 5 concludes with key insights and future directions.

POSITION-DEPENDENT MASS OSCILLATORS IN NATURE AND APPLICATIOS

A significant number of position-dependent mass (PDM) oscillatory systems are evident in nature, both in the plant and animal world. During motion, there is a change in the distribution of mass and the moment of inertia within the body as a consequence of position changes. These changes may be of structural kind or arise from variations within the material itself. For instance, in botany, tree or plant oscillation in the wind is a classic example of such systems. Wind induces periodic bending of tree branches or stems. As water content and internal pressure vary along the length, the effective mass and inertia change with position (Niklas, 1992; Vogel, 2012). Large plant leaves and flower stalks, such as those in banana plants or sunflowers, exhibit PDM behavior under wind loading, where the structural response and effective mass vary with position (Spatz & Speck, 2002).

Locomotion in living beings is also frequently accompanied by changes in the effective mass of the body. In undulatory swimmers such as eels and lampreys, the body mass distribution changes dynamically along the spine due to fluid-structure interactions and muscle activation patterns (Tytell et al., 2010). These organisms exhibit traveling wave oscillations with position-dependent inertia and damping, leading to highly efficient propulsion strategies in aquatic environments. Snakes also exhibit periodic body deformation during locomotion, changing their mass distribution and moment of inertia with each undulatory wave (Jayne, 1986).

Additional examples of PDM oscillators can be found in the flexible tail beats of swimming fish, such as tuna and sharks (Shadwick & Gemballa, 2005), and the morphing wings of birds like swifts, which dynamically adjust wing shape to control glide performance (Lentink et al., 2007).

Insects, including flies, bees, and moths, rely on wing flapping for flight. During flapping, the wings dynamically change their shape – an active deformation process – altering the mass distribution in each stroke (Combes & Daniel, 2003). The flexible wing structure modifies its effective mass and stiffness throughout the cycle, enabling agile and controlled flight maneuvers.

Furthermore, insect antennae, such as those of stick insects and moths, act as elastic oscillators with position-dependent effective mass due to hemolymph redistribution and dynamic shape changes (Dürr & Ebeling, 2005).

Propulsion mechanisms in aquatic organisms further exemplify PDM oscillators. Jellyfish contract and expand their bell to swim. Contraction expels water, reducing internal mass, and subsequent refilling restores it, resulting in a fluid-mass cycle that changes the inertia of the system (Gemmell et al., 2015). Similarly, octopuses and squids employ jet propulsion: they fill a cavity with water, increasing internal mass, and then eject it to generate thrust. Here, the system mass varies with internal volume, and the cavity shape directly affects the total system mass (Anderson & Grosenbaugh, 2005).

Particularly interesting is the occurrence of PDM oscillators within the human body. Heart valves and chambers act as oscillators with position-dependent mass during the cardiac cycle. Blood redistribution leads to changes in the effective mass of moving walls with each heartbeat (Chung & Im, 2012). As the heart fills and empties, its stiffness and effective mass vary, influencing overall hemodynamics. In voice production, the vocal folds oscillate dynamically. During phonation, their mass distribution changes with shape and tension, functioning as nonlinear oscillators with effective mass varying along the vocal fold length (Titze, 2008; Cvetićanin, 2012). This complexity contributes to the richness and variability of human voice production.

Finally, these principles are also evident in limb movements. During walking or running, human and animal limbs exhibit periodic motion, with changes in mass distribution and effective moment of inertia driven by muscle contraction, fluid shifts, and joint angles (Winter, 2009). These variations are essential for understanding the biomechanics of locomotion and for improving performance and rehabilitation strategies.

What is common to all these examples is that they represent oscillators in which mass and/or moment of inertia change depending on position. Consequently, to achieve the most accurate simulation of such systems, it is essential to develop precise models incorporating position-dependent mass (PDM). Such models should consider as many relevant factors as possible, including material heterogeneity, structural geometry, and fluid interactions. Beyond mimicking natural systems, PDM models hold potential for diverse engineering and medical applications. They can support the development of medical devices for organ treatment, aid in the creation of artificial organs, and contribute to the production of artificial voices. Further research in this field can advance both our understanding of biological systems and the design of innovative bioinspired technologies.

MODEL OF POSITION-DEPENDENT MASS OSCILLATOR

Let us model the oscillator as a spring – particle system. The particle has the position dependent mass (PDM) $m(x)$, which depends on the displacement coordinate x. Including the specificity of structure and material properties of the system, the function $m(x)$ may describe any of aforementioned cases. For particle with PDM the kinetic energy is

$$E_k = \frac{1}{2} m(x)\dot{x}^2 \tag{1}$$

where $\dot{x}$. is the velocity. Thus, kinetic energy of PDM particle is not only the function of the velocity as is the case for particle with constant mass, but also of the displacement. Properties of the kinetic energy (1) are discussed in paper of Mustafa (2012) and Chargui (2019).

For the elastic spring with potential $V(x)$ the Lagrangian L of the oscillator is (Tkachuk & Voznyak, 2015; Mustafa & Algadhi, 2019; Mustafa, 2020; Biswas, 2020)

$$L = \frac{1}{2} m(x)\dot{x}^2 - V(x) \tag{2}$$

Using the Lagrange formalism $\frac{d}{dt}\frac{\partial L}{\partial x} - \frac{\partial L}{\partial x} = 0$ the equation of motion follows as

$$m(x)\frac{d^2x}{dt^2} + \frac{1}{2}\frac{dm(x)}{dx}\left(\frac{dx}{dt}\right)^2 + \frac{dV(x)}{dx} = 0 \tag{3}$$

Expression (3) is a special type of Liénard equation (Liénard, 1928; Jordan & Smith, 2007) with quadratic damping (Ruby et al., 2015; Rath et al., 2017). The effect of quadratic damping on the oscillatory motion was discussed for a long time (Tiwari et al., 2013). Verhulst (1996) gave the general treatment of quadratic damping and suggested the consideration of the Liénard equation within the framework of dynamical system. In the papers of Narayanan & Sekar (1996) and Bishop & Clifford (1996) not only the unforced but also the forced periodical oscillators are considered. The focus is on periodic response and stability of such systems. It is shown how quadratic damping alters response amplitudes and frequency behavior. Zhou & Zhang (2007) extended the research considering chaos in the Liénard system. Bifurcation structures and the route to chaos in these systems is discussed.

For (1), significant number of analytic solving procedures are developed. Let us mention some of them: the multiple scale method (Nayfeh & Mook, 1979), perturbation techniques (Chatterjee & Mallik, 1994), harmonic balance method (Bishop & Clifford, 1996). Main disadvantage of the mentioned procedures is that they are based on the harmonic solution of the linear oscillator, in spite of the fact that the system is nonlinear. In Cvetićanin et al. (2025), modification of methods is done and the solution is assumed to be the perturbed version of the exact solution of the truly nonlinear oscillator. The procedure is suggested for various mass functions.

Harmonic like PDM oscillator

According to the formulation of the potential energy of the linear spring in the system with constant mass $V=k\frac{x^2}{2}$ where the stiffness coefficient k is defined as the product of the natural frequency ω^2 of the oscillatory system and mass, the potential energy of the PDM oscillator is (Rath et al., 2021)

$$V(x)=\frac{m(x)\omega^2 x^2}{2} \tag{4}$$

Substituting (4) into (3) the equation of motion is

$$\ddot{x}+\omega^2 x+\frac{dm(x)}{dx}\frac{1}{m(x)}\left(\frac{\dot{x}^2}{2}+\frac{\omega^2 x^2}{2}\right)=0 \tag{5}$$

As the first two terms in equation correspond to harmonic oscillator and the last two represent the perturbation of the oscillator with constant mass, the equation (5) is named '*harmonic like PDM oscillator*' (Carinena et al., 2004; Asad et al., 2020, Takou et al., 2025a). Various mass variations are considered (Dong et al., 2007; Costa-Filho, 2011; Dong et al., 2022) and their effect on motion is analyzed. Ghosh and Modak (2009) discussed the role of PDM-symmetry on trajectory of motion and Khlevniuk (2018) established a geometric interpretation of the motion of a classical particle with PDM in a harmonic potential. Rath et al. (2021) and Jafarov & Nagiyev (2023) extended the research by formulating oscillators with position-dependent finite symmetric decreasing and increasing mass. Classical phase portraits of the systems were analyzed using analytical approaches. Takou et al. (2025b) introduced a thermodynamic analysis of a harmonic oscillator with position-dependent mass and explored also the statistical properties and thermodynamic quantities of the system.

Mathews and Lakshmanan (1974) considered the so called '*Mathews-Lakshmann oscillator*' with PDM of Lorentzian profile type (Mathews & Lakshmanan, 1975). This equation has the exact analytic solution in the form of the cosine trigonometric function (Lakshmanan & Chandrasekar, 2013; Karthiga et al., 2017; Santos & González-Borrero, 2023).

Truly nonlinear like PDM oscillator

In general, let us assume the spring to have nonlinear elastic property. The potential energy for the spring is (Cveticanin, 2018)

$$V(x)=\frac{m(x)\omega_\alpha^2|x|^{\alpha+1}}{\alpha+1} \tag{6}$$

where $\alpha \geq 1 \wedge \alpha \cup R$ (integer or noninteger) is the nonlinearity order of the oscillator and ω_α^2 is a constant. Substituting ( ) into ( ) – ne znam šta treba da bude u zagradama (Lj.Tubić) the equation of motion is

$$\ddot{x}+\omega_\alpha^2 x|x|^{\alpha-1}+\frac{dm(x)}{dx}\frac{1}{m(x)}\left(\frac{x^2}{2}+\omega_\alpha^2\frac{x|x|^\alpha}{\alpha+1}\right)=0 \tag{7}$$

Comparing (7) with the oscillator with constant mass

$$\ddot{x}+\omega_\alpha^2 x|x|^{\alpha-1}=0 \tag{8}$$

the difference is evident. The additional terms in (7) are the products of PDM function and its position derivative. As the equation (7) represents the perturbation of (8), i.e. of the truly nonlinear oscillator (Mickens, 2010), the equation (7) is usually called '*truly nonlinear like PDM oscillator*'.

Analyzing (7) it is obtained that it has the first integral

$$m(x)\left(\frac{x^2}{2}+\omega_\alpha^2\frac{x|x|^\alpha}{\alpha+1}\right)=K=const. \tag{9}$$

The first integral (9) is of energy type where both energies (kinetic and potential) are mass dependent. For initial conditions

$$x(0)=A,\quad \dot{x}(0)=0 \tag{10}$$

where A = const. the constant is $K=m(A)\frac{A^{\alpha+1}}{\alpha+1}$ and the energy integral is

$$m(x)\left(\frac{x^2}{2}+\omega_\alpha^2\frac{x|x|^\alpha}{\alpha+1}\right)=m(A)\frac{A^{\alpha+1}}{\alpha+1} \tag{11}$$

i.e.

$$\dot{x}=\pm\sqrt{2\left(\frac{m(A)}{m(x)}\frac{A^{\alpha+1}}{\alpha+1}-\omega_\alpha^2\frac{|x|^{\alpha+1}}{\alpha+1}\right)} \tag{12}$$

Expression (11) i.e. (12) describes the orbital periodic motion with amplitude A. Using (12) the period of vibration is

$$T=\sqrt{\frac{2(\alpha+1)}{m(A)A^{\alpha+1}}}\int_{-A}^{A}\frac{dx}{\sqrt{\frac{1}{m(x)}-\omega_{\alpha}^{2}\frac{|x|^{\alpha+1}}{\alpha+1}\frac{\alpha+1}{A^{\alpha+1}m(A)}}} \tag{13}$$

Unfortunately, the exact analytic solution of (13) is not evident. Because of that the method for calculation of the approximate period of vibration is introduced.

Modified He's formulation

He, in his paper (He, 2006), introduced the formula for computing of the approximate frequency of vibration of the perturbed linear oscillator

$$\omega^{2}=\frac{\omega_{1}^{2}R_{2}(0)-\omega_{2}^{2}R_{1}(0)}{R_{2}(0)-R_{1}(0)} \tag{14}$$

where ω_1 is the exact frequency of the nonperturbed linear oscillator, $\omega_2=\omega$ is unknown frequency, and $R_1(0)$ and $R_2(0)$ are residuals obtained for the linear solution Acos(ωt) and corresponding frequencies.

For the truly nonlinear oscillator (8) and initial conditions (10) there is the exact solution $x = Aca(\alpha, 1, \omega t)$ in the form of the ca cosine Ateb function (Cveticanin, 2025) with exact frequency

$$\omega_{1}^{2}=\omega_{\alpha}^{2}A^{\alpha-1}\frac{\alpha+1}{2} \tag{15}$$

Substituting (15) into (14) and after some modification the approximate frequency of the ca Ateb function follows as

$$\omega=\sqrt{\omega_{\alpha}^{2}A^{\alpha-1}\frac{\alpha+1}{2}+\omega_{\alpha}^{2}\frac{A^{\alpha}}{2}\left(\frac{dm}{dx}\frac{1}{m}\right)_{0}} \tag{16}$$

Knowing that the Ateb function is periodic with period $B\left(\frac{1}{\alpha+1},\frac{1}{2}\right)$ where B is the complete beta function, the period of vibration is

$$T=\frac{2\mathrm{B}(\frac{1}{\alpha+1},\frac{1}{2})}{\omega}=\frac{2\mathrm{B}(\frac{1}{\alpha+1},\frac{1}{2})}{\sqrt{\omega_{\alpha}^{2}A^{\alpha-1}\frac{\alpha+1}{2}}}\frac{1}{\sqrt{1+\frac{A}{\alpha+1}\left(\frac{dm}{dx}\frac{1}{\mathrm{m}}\right)_{0}}} \tag{17}$$

In (17) the first term corresponds to the period of the truly nonlinear oscillator with constant mass, and the second is the correction term which depends on the PDM function and its position derivative. Analyzing the relation (17) it is obvious that the period is longer for decreasing PDM than for the oscillator with constant mass. In opposite, if mass is increasing the period is decreasing in comparison to the oscillator with constant mass. The higher is the mass increase, the period is shorter.

THEORETICAL FRAMEWORK AND APPLICATIONS OF PDM IN QUANTUM SYSTEMS

This section reviews the theoretical framework and practical implications of PDM models in quantum systems, particularly in the modeling of quantum oscillators. It highlights how the PDM formalism modifies standard quantum models, enabling more accurate descriptions of carrier behavior in heterostructures, quantum wells, and quantum wires. The theoretical foundations of PDM are examined, along with its significance in capturing the spatial variations of effective mass and its relevance in multiple application domains.

In engineered nanostructures such as heterostructures, quantum wells, and quantum wires, the spatial nonuniformity of material parameters necessitates refined theoretical models to enable accurate simulation and device optimization (Capasso et al., 1995; Zawadzki, 2005). The PDM approach thus represents an essential tool for describing quantum mechanical systems where the effective mass of charge carriers varies spatially (Roos, 1983; Bastard, 1988). Namely, recent advances in nanotechnology and semiconductor physics have increasingly emphasized the importance of spatially varying material parameters, especially the effective mass of charge carriers in semiconductor nanostructures. The PDM framework emerges as a crucial refinement to standard quantum models, accounting for the inhomogeneities encountered in real-world systems (Dekar et al., 1999; Plastino et al., 1999).

Furthermore, the analogy between mechanical and quantum PDM oscillators has been demonstrated (Carinena et al., 2007; Schulze-Halberg & Roy, 2016). The use of PDM models is widespread in quantum dynamics studies of low-dimensional systems (Costa Filho et al., 2011; El-Nabulsi, 2021a), particularly in the coherent state framework, where a specific quantum state closely approximates classical behavior. This is especially significant in the context of quantum dots, quantum wells, and other semiconductor nanostructures (Cruz & Ortiz, 2009; Costa et al., 2023).

PDM model and applications

In nanostructures such as quantum dots, quantum wells, and superlattices, the effective mass of carriers (electrons and holes) is not constant but varies with position. The PDM model captures this spatial variation, providing improved accuracy in describing energy states and carrier dynamics. Coherent states in these structures often serve to describe the quasi-classical behavior of confined carriers (e.g., in laser-based nanostructures and quantum emitters).

Key application areas include:

- *Semiconductor quantum dots*: Electrons are confined in all three spatial directions, with PDM models accurately describing energy levels and wave functions (Ghosh et al., 2016; El-Nabulsi, 2020a; 2020b; Sari et al., 2022).
- *Quantum wells*: Electrons can move freely in the plane of the well but are confined in one dimension, forming discrete, quantized energy levels (Harrison, 2005; Dekar et al., 1999).
- *Superlattices and nanoscale layers*: These structures exhibit engineered potential profiles relevant to nonlinear optics and coherent emission properties (Chen, 2008).

A heterostructure consists of multiple layers of dissimilar semiconductor materials, each with distinct electronic band structures (El-Nabulsi, 2021a; 2021b; Costa et al., 2021). These structures are typically fabricated using epitaxial growth techniques such as molecular beam epitaxy (MBE) or metal-organic chemical vapor deposition (MOCVD), allowing atomic-level control of layer thickness and composition (Chen, 2008).

Quantum wells arise when a thin semiconductor layer with a lower bandgap is confined between layers of higher bandgap material (Harrison, 2005). This structure creates a potential well that confines carriers in one spatial dimension, leading to discrete energy levels (Dekar et al., 1999). Quantum wires extend this confinement to two dimensions, allowing motion only along a single axis and are typically fabricated using electron-beam lithography and anisotropic etching (Chen, 2008). While idealized models assume a constant effective mass, real-world systems exhibit spatial variations due to changes in alloy composition, band structure, and quantum confinement effects (Roos, 1983; Bastard, 1988; Zawadzki, 2005). Accurate modelling thus requires incorporating PDM effects to account for variations in energy spectra, carrier mobility, and tunneling rates (Serra & Lipparini, 1997; Koç & Koca, 2003).

Quantum Oscillators with Position-Dependent Mass

The PDM oscillator approach is essential for simulating devices where material inhomogeneity strongly affects carrier motion (Serra & Lipparini, 1997; Mottaghizadeh & Sadeghi, 2020). These oscillators extend classical harmonic and inharmonic oscillator models by incorporating mass variation, leading to modified energy levels and wavefunction profiles (Dutra & Almeida, 2000; Bagchi et al., 2005; Bagchi et al., 2012; Costa et al., 2023; Takou et al., 2025b). These models are particularly relevant in:

- *Quantum wells and dots with smooth interfaces* (Harrison, 2005; Sayrac et al., 2025).
- *Semiconductor devices with spatially graded composition or doping* (Bastard, 1988).
- *Superlattices and nanostructures with engineered potential landscapes* (Christiansen & Lima, 2023; Lima & Christiansen, 2023).

For instance, high-electron-mobility transistors (HEMTs), utilized in high-frequency RF and satellite circuits, involve heterostructures such as GaAs/AlGaAs where PDM arises at interfaces between materials with different band-gaps (Roos 1983; Capasso et al., 1995; Peter, 2020). In scanning tunneling microscopy (STM) systems, the tip interacts with layered quantum materials where electronic properties–and effective mass–vary spatially, influencing tunneling probability and imaging resolution (Chen, 2008; Jaradat et al., 2024a; 2024b). Resonant tunneling diodes (RTDs) utilize double-barrier quantum wells, where varying effective mass influences tunneling rates and energy levels (Bagchi et al., 2005; Bagchi et al., 2012). Quantum cascade lasers (QCLs), employed in spectroscopy, gas sensing, and medical diagnostics, consist of repeated quantum wells with alternating materials, resulting in PDM across the layers (Capasso et al., 1995; Callebaut & Hu, 2005). Graded-index waveguides, quantum dot LEDs, and solar cells similarly exhibit PDM effects due to compositionally graded materials (Bastard, 1988; Mottaghizadeh & Sadeghi, 2020; Ullah & Ullah, 2020; Sayrac et al., 2025).

The PDM effects also manifest in strongly inhomogeneous plasmas, where spatially varying electric and magnetic fields modulate effective mass, impacting Langmuir wave propagation and particle trapping (Christiansen & Lima, 2023; El-Nabulsi & Anukool, 2022). These systems often display nonuniform energy spacing, asymmetric tunneling, and position-sensitive resonance phenomena, underscoring the importance of PDM oscillator models in understanding such effects (Quesne, 2023).

CONCLUSION

This review has provided a thorough overview of position-dependent mass (PDM) models and their theoretical frameworks, highlighting their essential role in understanding natural and engineered oscillatory systems. From undulatory swimmers and flexible plant structures to insect flight mechanisms, PDM models accurately capture dynamic mass variations crucial to locomotion, fluid–structure interactions, and adaptive responses in biological systems.

In engineering and quantum domains, PDM models have proven vital for the design and optimization of advanced electronic, photonic, and nano-scale devices–including quantum cascade lasers, high-electron-mobility transistors, and scanning tunneling microscopy. Their ability to integrate mass variation and motion dynamics makes them indispensable in both fundamental and applied research.

The broad relevance of PDM models in biology, engineering, and quantum systems underscores their versatility in capturing the complex dynamical behavior of oscillators with position-dependent inertia. This unified framework holds promise for guiding the development of biomimetic devices, soft robotics, and adaptive materials inspired by natural strategies for movement and stability.

Future research should aim to refine these models to incorporate nonlinear effects, anisotropic material behavior, and complex damping mechanisms while validating their predictions experimentally in both natural and engineered systems. Integrating PDM-based models into next-generation quantum devices, photonic systems, and plasma-based technologies offers substantial opportunities for innovation and performance enhancement. Ultimately, PDM-inspired oscillators bridge the gap between natural insights and engineering solutions, paving the way for transformative applications across diverse scientific and technological fields.

OUTLOOK AND FUTURE WORK

The advancement of accurate PDM models for living systems demands interdisciplinary collaboration among biomechanics, physiology, fluid dynamics, and computational modeling. Future research should prioritize the refinement of mathematical formulations for PDM systems–particularly in the context of complex biological tissues and fluid–structure interactions–supported by high-fidelity experimental data from time-resolved MRI, high-speed imaging, and other advanced techniques.

Furthermore, leveraging machine learning and optimization algorithms will be instrumental in identifying key parameters that govern PDM system behavior, facilitating more robust and precise control strategies in real-world applications. Integrating these data-driven approaches with theoretical models promises to accelerate the development of predictive and adaptive PDM frameworks.

Looking ahead, the combination of theory, experimentation, and computational simulation will be crucial to unlocking new insights into these complex yet elegant natural phenomena. Such efforts will not only enhance our understanding of biological and quantum systems but also inspire the next generation of bioinspired, energy-efficient, and adaptable engineered devices that fully exploit the rich dynamical properties of PDM-based oscillators.

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ПРЕГЛЕД ПРИРОДОМ ИНСПИРИСАНИХ МОДЕЛА ОСЦИЛАТОРА СА ПРОМЕНЉИВОМ МАСОМ ПРИМЕЊЕНИХ У КВАНТНИМ СИСТЕМИМА

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РЕЗИМЕ: Преглед истражује концепт осцилатора са масом зависном од положаја (МЗП) препознат код природних система а који је применљив код квантних уређаја. Биолошки осцилаторни системи показују динамичко прилагођавање расподеле масе кретању и положају. Промена ефективне масе и момента инерције дешава се код савијања стабљике биљке, код вијугања тела рибе при пливању, код крила при летењу птица али и инсеката, код удова човека и животиња при корачању и трчању. Рад низа унутрашњих органа у телу човека омогућен је и пропраћен осцилаторним кретањем уз промену масе (рад срчаног мишића, треперење гласне жице). Инспирисани овим природним феноменима сачињени су модели осцилатора са ефективном масом која је променљива, и функција положаја. Физички модел осцилатора који симулира кретања састоји се од еластичног елемента за који је везано тело променљиве масе. У општем случају математички модел система је Лиенардова једначина са квадратном функцијом брзине. У раду су приказане специфичности овог осцилатора које се односе на тип кретања, период и амплитуде осциловања. Ови модели нуде свестран теоријски оквир за разумевање осцилатора са просторно променљивом инерцијом, обухватајући сложене интеракције између структуре и кретања. У раду је, користећи карактеристике механичког модела, повучена паралела између природних система и инжењерских квантних уређаја. Преглед обухвата теоријски развој МЗП модела, њихове механичко-квантне аналогије и њене примене. У квантним системима, нарочито у полупроводничким наноструктурама као што су квантне јаме, жице и тачке, просторне варијације ефективне масе носилаца наелектрисања проистичу из композиционих нехомогености и структурних градијената. МЗП модели омогућују усавршавање традиционалног приступа квантне механике, побољшавајући тачност предвиђања нивоа енергије и динамике носилаца у овим системима. Такође, ови модели чине основу за пројектовање и рад напредних електронских и фотонских уређаја попут квантних каскадних ласера, транзистора са великом покретљивошћу електрона и скенирајуће тунелске микроскопије. Повезујући природне и инжењерске перспективе, овај рад указује на велики потенцијал МЗП осцилатора као оквира за вођење иновација у биолошки инспирисаним технологијама, адаптивним материјалима и квантним уређајима следеће генерације. Будућа истраживања треба да укључе нелинеарне ефекте, анизотропне материјале и стратегије оптимизације засноване на подацима, како би се додатно усавршили МЗП модели и искористио њихов пуни технолошки потенцијал.

КЉУЧНЕ РЕЧИ: маса зависна од положаја (МЗП); модели инспирисани природом; нелинеарни осцилатори; квантни системи